Methods of Decision Making and Fuzzy Sets in the Selection of the Optimal Location for a Metro Station

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Abstract — **The paper is devoted to solving the problem of selecting the optimal location of a metro station in a microdistrict with established urban development and infrastructure. The problem is solved using a combined method based on the method of multi-criteria selection of alternatives and fuzzy sets theory. As criteria are considered both absolute, such as distance to the surface, the presence of a ground lobby, distance to the center of passenger traffic, etc., and relative, such as the convenience of the location of the exit for different categories of population living in the territory of the microdistrict. Calculating weighted degrees of preference for a relative criterion for all options is a separate sub-task and uses a number of absolute indicators for several criteria for different categories of the population. The final selection of an alternative is made in two options: for the case of equilibrium and for non-equilibrium attributes.**

I. Introduction

Over the last decades, there has been a steady tendency towards a constant increase in the population of megacities all over the world. In order to ensure the comfort level of residents, it is necessary to continuously develop urban infrastructure, including transport infrastructure. For this purpose, new stations are being built in cities with a metro system. The metro is a highly complex technical and strategic facility that operates underground. During the design and construction of metro lines, extensive engineering and geological studies are carried out, which may result in several options regarding the possible placement of stations on the line, or several options for the location of exits from the station to the surface. Often, there is already some infrastructure at the proposed station construction site, including buildings of various purposes, transport routes, and open spaces — especially when creating new stations or transfer stations within existing urban development. In such a situation, the choice of the location of the surface exit of a new metro station, from among the technically feasible options, is, in fact, a multi-criteria optimisation problem. The optimality of the exit location is determined by many criteria — both objective, such as the reduction of construction costs or maximization of station capacity, and subjective, from the point of view of different categories of the population living in a given neighbourhood.

In this paper, a combined method using fuzzy sets [1],[2] and decision-making methods [3],[4] is proposed to find the optimal location of the exit from the station to the surface. It is based on a fuzzy multi-criteria analysis of options on the basis of pairwise comparisons [5] of subjective judgments numerically evaluated by experts on a nine-point Saaty scale [3], with their subsequent agreement. The analysis of variants is based on available linguistic information about their quality, which is most suitable for experts. To calculate the vector of weighted degrees of preference for exit locations, a methodology based on solving the problem of partitioning into trade zones under fuzzy conditions by Y. Leung [6] is applied. Based on the method of determining the center of gravity of the physical model of the distribution system [7], we can separately calculate the center of gravity of the passenger flow within the city microdistrict. The best option is considered to be the location of the station with the minimum distance to the center of gravity. The Bellman-Zadeh principle [8] is used to determine the best location option simultaneously for all criteria, taking into account the importance of each criterion. Various approaches to decision-making problems are described in [9], [10], [11], [12], [13].

II. Fuzzy Sets and Relations

The following concepts of fuzzy set theory are used hereafter [1], [2], [11].

A fuzzy set A is defined as a set of ordered pairs of the form $\left\{\frac{\mu_A(u)}{u}\right\}$, $u \in U$, where $\mu_A(u) \in [0,1]$ – the membership function of the fuzzy set A, U – the *universe* of values. For each $u \in U$ the membership function $\mu_A(u)$ determines the degree to which an element u belongs to the set A. If $\forall\, u \in U$ such that $\mu_A(u) = 0$, then the fuzzy set A is called an empty set. If $U = \{0, 1\}$, the fuzzy set A can be considered as a regular, clear set. If A and B are defined on the same universe, B is a *subset* of A ($B \subseteq A$) if and only if the following is true:

$$\mu_B(u) \le \mu_A(u), \forall\, u \in U.$$

The α-level set of a fuzzy set A is the set A_α defined by the formula:

$$A_\alpha = \{u | u \in U,\ \mu_A(u) \ge \alpha\}, \alpha > 0. \quad (2.1)$$

Let A and B be fuzzy sets defined on the universe U. Then A and B are *equal* if $\forall\, u \in U$ $\mu_B(u) = \mu_A(u)$. The union of fuzzy sets A and B is a fuzzy set $A \cup B$ with a membership function

$$\mu_{A\cup B}(u) = \max(\mu_A(u), \mu_B(u)), \forall\, u \in U.$$

The intersection of fuzzy sets A and B is a fuzzy set $A \cap B$ with a membership function

$$\mu_{A\cap B}(u) = \min(\mu_A(u), \mu_B(u)), \forall\, u \in U. \quad (2.2)$$

The *degree* of a fuzzy set A is a fuzzy set A^s with a membership function

$$\mu_{A^s}(u) = \mu_A{}^s(u), \forall\, u \in U, s > 0. \quad (2.3)$$

Let $A_1, \dots A_n$ – be fuzzy sets defined on the universe U. *Cartesian (direct) product* of fuzzy sets $A = A_1 \times \dots \times A_n$ is characterised by the membership function $\mu_A(u)$:

$$\mu_A(u) = \min\{\mu_{A_1}(u_1), \dots\ \mu_{A_n}(u_n)\},$$
$$u = (u_1, \dots u_n) \in \underbrace{U \times \dots \times U}_{n}.$$

Fuzzy binary relation R is a fuzzy set that is defined on the Cartesian product of the $U_1 \times U_2$ with a membership function μ_R: $U_1 \times U_2 \rightarrow [0, 1]$. The $\mu_R(u_1, u_2)$ is considered to represent the level of dependency between $u_1 \in U_1$ and $u_2 \in U_2$.

The α-level relation of the fuzzy relation R is the set R_α: $R_\alpha = \{(u_1, u_2) | (u_1, u_2) \in U_1 \times U_2, \mu_R(u_1, u_2) \geq \alpha\}, \alpha > 0$.

Let there be fuzzy relations R and Q on the set $U_1 \times U_2$. The intersection of fuzzy relations R and Q is called a fuzzy relation with the membership function $\mu_{R\cap Q}(u_1, u_2)$:

$$\mu_{R\cap Q}(u_1, u_2) =$$
$$= \min\{\mu_R(u_1, u_2), \mu_Q(u_1, u_2)\}, (u_1, u_2) \in U_1 \times U_2. \quad (2.4)$$

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$$= \max\{\mu_R(u_1, u_2), \mu_Q(u_1, u_2)\}, (u_1, u_2) \in U_1 \times U_2.$$

III. METHODS USED TO SOLVE THE TASK

Let P be a set of alternatives or options, and Y be a set of quantitative attributes by which the alternatives are evaluated. The problem is to find the best alternative based on attributes from the set Y. This problem belongs to the class of decision-making problems under fuzzy conditions [3],[4],[9]. For its solution, various approaches and methods are be combined: the method of multi-criteria analysis based on pairwise comparisons [5], which are carried out using a nine-point Saaty scale [3]; the Bellman-Zadeh principle for determining the best option [8]; the method of determining the center of gravity of the physical model of the distribution system [7]; the method of calculating weighted degrees of preference based on the solution of the problem of division into trade zones in fuzzy conditions, proposed by J. Leung [6].

A. Fuzzy Multi-Criteria Case Analysis Method

The method of multi-criteria analysis is described in detail in [5]. Unlike other known methods of multi-criteria analysis [13],[14], the method proposed by A.P. Rothstein and S.D. Shtovba requires neither quantitative assessment of attributes nor a scalarisation procedure, but only the use of linguistic information about the quality of alternatives in the form of pairwise comparisons. Its main feature is the ranking of the proposed alternatives using linguistic evaluations of individual features and determining the membership functions of qualitative feature evaluations using the method of pairwise comparisons on the Saaty scale [3]. The best alternative is selected based on the scheme proposed by Bellman and Zadeh in [8].

Let us consider a set of alternatives $P = \{p_1, \dots p_m\}$, proposed for selection, and perform a fuzzy multi-criteria analysis [5] of the set P by attributes from the set $Y = \{y_1, \dots y_l\}$, namely, arrange the alternatives from the set P in the order of preferences by attributes from Y.

B. Attributes as fuzzy sets

Let us consider the set of alternatives P as a universe on which we define l fuzzy sets with membership functions $\mu_{Q_i}(p_j) \in [0,1]$, $i = \overline{1,l}, j = \overline{1,m}$. The function $\mu_{Q_i}(p_j)$ characterises the degree of belonging of element p_j of the universal set of alternatives P to element y_l of the feature set Y. To construct the membership functions of these fuzzy sets, we use Saaty's method of pairwise comparisons.

For each attribute $y_l \in$ Y, we construct pairwise comparison matrices Q_i of the elements of the set of alternatives P, $i = \overline{1,l}$:

$$Q_i = \begin{matrix} & p_1 & p_2 & \cdots & p_m \\ p_1 & q_{11} & q_{12} & \cdots & q_{1m} \\ p_2 & q_{21} & q_{22} & \cdots & q_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_m & q_{m1} & q_{m2} & \cdots & q_{mm} \end{matrix}, \quad (3.1)$$

where q_{kj} $(k, j = \overline{1,m})$ – the degree of preference of alternative p_k to alternative p_j by attribute y_i, which is assessed by experts on a nine-point Saaty scale [3] (Table I).

TABLE I. SAATY NINE-POINT SCALE

Degree of preference q_{kj}	Qualitative assessment (comparison of p_k and p_j)
1	No advantage p_k over p_j
3	Weak advantage p_k over p_j
5	Substantial advantage p_k over p_j
7	A distinct advantage p_k over p_j
9	Absolute advantage p_k over p_j
2, 4, 6, 8	Interim comparative estimates

Each matrix of pairwise comparisons Q_i $i = \overline{1,l}$, is diagonal, i.e. $q_{kk} = 1$, $k = \overline{1,m}$, inversely symmetric, i.e. $q_{kj} = 1/q_{jk}$, $(k, j = \overline{1,m})$.

In this case, it is true [3] that a positive-definite, reciprocally symmetric matrix of pairwise comparisons is consistent, if the dimensionality of the matrix and its maximum eigenvalue are equal: $\lambda_{max} = m$. In the following, we will consider only consistent matrices of pairwise comparisons, which are compiled by experts with maximum confidence in their estimates.

To each feature $y_i \in Y$, $k, j = \overline{1,l}$, we assign a fuzzy set $\widetilde{Q_\iota}$, based on a matrix (3.1):

$$\widetilde{Q_\iota} = \left\{\frac{\mu_{Q_i}(p_1)}{p_1}, \dots \frac{\mu_{Q_i}(p_m)}{p_m}\right\}, \quad (3.2)$$

where $\mu_{Q_i}(p_j)$ – is the degree of membership of $p_j \in$ P in the fuzzy set $\widetilde{Q_\iota}$.

The paper [5] describes a method of constructing the membership function of a fuzzy set based on the matrix of pairwise comparisons of alternatives. According to this method, the degrees of membership are equal to the corresponding elements of the eigenvector $Z = \{z_1, \dots z_m\}$ of the matrix Q_i (3.1), corresponding to the maximum eigenvalue. We will use this method to find the degrees of membership $\mu_{Q_i}(p_j)$, $i = \overline{1,l}, j = \overline{1,m}$. Solve the system of equations:

$$\begin{cases} Q_i \cdot Z = \lambda_{max} \cdot Z, \\ z_1 + z_1 + \dots + z_m = 1, \end{cases} \tag{3.3}$$

where λ_{max} – maximum eigenvalue of the matrix Q_i.

As a result of solving the system (3.3) we obtain the following degrees of membership of the fuzzy set $\widetilde{Q_i}$:

$$\mu_{Q_i}(p_j) = z_i,\ i = \overline{1,l}, j = \overline{1,m}.$$

The found membership functions $\mu_{Q_i}(p_j)$ define the fuzzy sets $\widetilde{Q_\iota}$ (3.2). The alternative p_j, which has the maximum value $\mu_{Q_i}(p_j)$, is considered the best one according to the feature on attribute y_i.

C. Equilibrium and non-equilibrium attributes

Based on the Bellman-Zadeh scheme [8] the best alternative is the one that is best simultaneously on all attributes $y_1, \dots y_l$. Therefore, we will look for a fuzzy set $\tilde{G}$ as the intersection of fuzzy sets $\widetilde{Q_\iota}$ ($i = \overline{1,l}$), corresponding to different attributes y_i. The alternative that is best in all attributes will have the maximum value $\mu_{\tilde{G}}$.

Attributes $y_1, \dots y_l$ can have weights with which they enter the fuzzy set $\tilde{G}$, if the weights are the same, they are taken equal to 1, in this case the attributes $y_1, \dots y_l$ will be considered as equilibrium. If the weights are different (we denote them by s_i), and $s_1 + \dots + s_l = 1$, attributes $y_1, \dots y_l$ will be considered non-equilibrium and their weights are taken into account when constructing $\tilde{G}$.

Let us define a fuzzy set $\tilde{G}$ for equilibrium features by generalizing the expression (3.2):

$$\tilde{G} = \widetilde{Q_1} \cap \dots \cap \widetilde{Q_l} = \left\{ \frac{\min\limits_{i=1,l}[\mu_{Q_i}(p_1)]}{p_1}, \dots \frac{\min\limits_{i=1,l}[\mu_{Q_i}(p_m)]}{p_m} \right\}. \tag{3.4}$$

In the case of non-equilibrium attributes, it is necessary to take into account the weights of attributes (preference coefficients), i.e., when using the methodology of decision-making based on preference coefficients, there should be an increase in the difference between alternatives on the most important attributes, and, on the contrary, a reduction in the difference on the least important ones. Therefore, the set $\tilde{G}$ is defined by the intersection of fuzzy sets $\widetilde{Q_\iota}$ in the corresponding degrees s_i, $i = \overline{1,l}$.

The weight s_i is calculated using the method based on the order scale proposed in [15]. According to this method, each attribute y_i is assigned a score A_{y_i} on a scale, that uses integers consecutively from 1 to l, in case there are no features with equal importance among the studied features. If there are, they are grouped into sets with the same rating. All attributes are evaluated on a scale from 1 to $(l - \tilde{\imath} + \tilde{\jmath})$, where $\tilde{\imath}$ – is the total number of attributes with the same degree of importance in all groups, and $\tilde{\jmath}$ – is the number of these groups. For example, if we consider 7 attributes (l = 7), of which y_1, y_2, y_3 have the same degree of importance among themselves, y_4, y_5 – have different degrees of importance, and y_6, y_7 – have the same degree of importance. In this case $\tilde{\imath}$= 5, $\tilde{\jmath}$ = 2, and we rate the attributes on a scale of 1 to 4.

Thus, in both cases, the weight s_i is found by dividing each score A_{y_i} of attribute y_i by the sum of all the scores obtained, $i = \overline{1,l}$:

$$s_i = A_{y_i} / \textstyle\sum_{y_l} A_{y_i}. \tag{3.5}$$

Taking into account the weights and (3.3), expression (3.4) takes the form:

$$\tilde{G} = \widetilde{Q_1^{s_1}} \cap \dots \cap \widetilde{Q_l^{s_l}} = \\ = \left\{ \frac{\min\limits_{i=1,l}[\mu_{Q_i}(p_1)]^{s_i}}{p_1}, \dots \frac{\min\limits_{i=1,l}[\mu_{Q_i}(p_m)]^{s_i}}{p_m} \right\}. \tag{3.6}$$

where s_i– weight of the attribute y_i, $s_1 + \dots + s_l = 1$.

As the degree s_i increases, the difference between the elements of the fuzzy set becomes greater.

The best alternative p_j ($j = \overline{1,m}$), will be the one that has the highest degree of belonging to the set $\tilde{G}$ (3.6) for non-equilibrium features or to the set (3.4) in the case of equilibrium features.

The given method of multi-criteria analysis of alternatives based on pairwise comparisons [5] allows making decisions under fuzzy conditions, which is most convenient for experts. The applied Bellman-Zadeh scheme [8] ensures the selection of an alternative that simultaneously satisfies all attributes to the greatest extent.

D. Method for determining the center of gravity of a physical model of a distribution system

Consider a model of some city district with a given road infrastructure. It is necessary to determine the coordinates of location of the distribution center (e.g., warehouse complex) of this district with respect to a set of objects (consumers), $D = \{d_1, \dots d_m\}$.

The main idea of the method proposed in [7] is to locate the distribution center at a point that represents the center of gravity, in such a way that the sum of distances from this point to all consumers is minimal.

The method has a limitation: in the considered model there must be a sufficiently developed road system, since the distance between the point of material flow consumption and the location of the distribution center is calculated as a straight line. Let us apply this method to determine the center of gravity of passenger flow.

Let us introduce a coordinate system with $\tilde{X}$ and $\tilde{Y}$ axes, transfer the contour of the given district to it and calculate the coordinates $(\tilde{x}_j, \tilde{y}_j)$ $(j = \overline{1,m})$ for consumer objects $\{d_1, \dots d_m\}$. Let cargo turnover Γ_j (the amount of cargo delivered to each object per unit time) be given for each object d_j, $j = \overline{1,m}$.

Using the following formulas, we determine the coordinates $(\tilde{X}_{center}, \tilde{Y}_{center})$ of the location of the center of gravity of the freight turnover of the proposed city area model:

$$\tilde{X}_{center} = \frac{\sum_{j=1}^{m} \Gamma_j \cdot \tilde{x}_j}{\sum_{j=1}^{m} \Gamma_j}, \tilde{Y}_{center} = \frac{\sum_{j=1}^{m} \Gamma_j \cdot \tilde{y}_j}{\sum_{j=1}^{m} \Gamma_j}. \quad (3.7)$$

where $\tilde{x}_j, \tilde{y}_j$ – coordinates of the j-th object.

In real location, it is not always possible to locate a distribution center, such as a warehouse complex, at the point with the coordinates of the center of gravity of cargo turnover, then, it is located as close to it as possible, taking into account other factors.

E. Method of calculating weighted degrees of preference

The calculation of weighted degrees of preference is performed based on the method of partitioning into trade zones under fuzzy conditions, which was proposed by J. Leung in [6].

Let us consider a microdistrict with residential complexes H_k $(k = \overline{1,K})$. It is necessary to solve the problem of making a decision on choosing the most convenient location of a new object for the population of these residential complexes from the proposed options p_j $(j = \overline{1,m})$. Namely, to calculate the vector of weighted degrees of preference, the element of the vector, that has the highest value will correspond to the most convenient location of the new object.

The population of the microdistrict is divided into categories according to predetermined attributes, depending on the conditions of the problem, i.e., those features that the population should have to make a decision on the choice of location, and the attributes on which we perform the division are considered as fuzzy sets with appropriate membership functions. For example, if we divide the population by age, we can obtain the following population categories:

- young;
- middle-aged;
- elderly.

Considering these categories as terms of a fuzzy linguistic variable with their membership functions [11], it is possible to determine by age the degree to which an individual belongs to a category according to the rule of the dominant alternative [16].

Let $X = \{x_1, \dots x_N\}$ – a set of population categories, that live in these residential complexes; $G = \{g_1, \dots g_t, r\}$ – a set of criteria characterizing the *most convenient* location of some new object, and each criterion is defined by a fuzzy set and its corresponding membership function.

The $r \in G$ criterion demonstrates the effect of distance from residential complex H_k $(k = \overline{1,K})$, to a possible location p_j $(j = \overline{1,m})$, on one or the other category of population x_n $(n = \overline{1,N})$. For example, the attitudes towards the *Distance* criterion will be different between the 'young' and 'elderly' population categories. Let us also assume that the preferences of the population are not related to a particular residential complex, but only to the population category to which the individual belongs.

Let the matrix Ω settlement by residential complexes be given H_k $(k = \overline{1,K})$:

$$\Omega = \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \overset{\begin{matrix} H_1 & \dots & H_K \end{matrix}}{\begin{pmatrix} \Omega_{11} & \dots & \Omega_{1K} \\ \Omega_{21} & \dots & \Omega_{2K} \\ \vdots & \ddots & \vdots \\ \Omega_{N1} & \dots & \Omega_{NK} \end{pmatrix}}, \quad (3.8)$$

where the element Ω_{nk} – is the number of people living in the residential complex H_k $(k = \overline{1,K})$ and corresponding to the population category x_n $(n = \overline{1,N})$.

Based on the experts' assessments on a nine-point Saaty scale [3] we construct the following matrix of consistent pairwise comparisons B_n for each population category x_n $(n = \overline{1,N})$:

$$B_n = \begin{matrix} g_1 \\ g_2 \\ \vdots \\ g_t \\ r \end{matrix} \overset{\begin{matrix} g_1 & g_2 & \dots & g_t & r \end{matrix}}{\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1t} & b_{1(t+1)} \\ b_{21} & b_{22} & \dots & b_{2t} & b_{2(t+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{t1} & b_{t2} & \dots & b_{tt} & b_{t(t+1)} \\ b_{(t+1)1} & b_{(t+1)2} & \dots & b_{(t+1)t} & b_{(t+1)(t+1)} \end{pmatrix}}, \quad (3.9)$$

where the element $b_{u\tilde{u}}$ – the degree of advantage of criterion g_u over criterion $g_{\tilde{u}}$ $(u, \tilde{u} = \overline{1, t+1})$, and $g_{t+1} = r$.

Let us define a fuzzy binary relation R on the universe $X \times G$ with a membership function $\Phi_R(x_n, g_u)\colon X \times G \to [0, 1]$. For each $x_n \in X$ and each $g_u \in G$ the function $\Phi_R(x_n, g_u)$ is the degree of relative importance of criterion g_u as judged by an individual in category x_n in determining his or her preference for the location of the object.

The matrix of fuzzy binary relation R has the following form:

$$R = \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \overset{\begin{matrix} g_1 & g_2 & \dots & g_t & r \end{matrix}}{\begin{pmatrix} \Phi_R(x_1, g_1) & \Phi_R(x_1, g_2) & \dots & \Phi_R(x_1, g_t) & \Phi_R(x_1, r) \\ \Phi_R(x_2, g_1) & \Phi_R(x_2, g_2) & \dots & \Phi_R(x_2, g_t) & \Phi_R(x_2, r) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi_R(x_N, g_1) & \Phi_R(x_N, g_2) & \dots & \Phi_R(x_N, g_t) & \Phi_R(x_N, r) \end{pmatrix}}, \quad (3.10)$$

where the rows are eigenvectors corresponding to the maximum eigenvalues of the matrix of pairwise comparisons B_n $(n = \overline{1,N})$ (3.9), with the largest value in a row corresponds to the most important criterion.

Again, we will use the method of Saati [3] to determine the advantage of location p_j over location p_θ by the criterion g_u $(j, \theta = \overline{1,m}, u = \overline{1,t})$ from the local resident's point of view — with the exception of the *Distance* r criterion. Assuming that the experts' assessments are consistent, we obtain the following matrices of pairwise comparisons C_u $(u = \overline{1,t})$:

$$C_u = \begin{matrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{matrix} \overset{\begin{matrix} p_1 & \dots & p_m \end{matrix}}{\begin{pmatrix} p_{11} & \dots & p_{1m} \\ p_{21} & \dots & p_{2m} \\ \vdots & \ddots & \dots \\ p_{m1} & \dots & p_{mm} \end{pmatrix}}, \quad (3.11)$$

For the *Distance* r criterion, we similarly separately compile consistent pairwise comparison matrices C^k, where k – is an index that indicates in which residential complex H_k $(k = \overline{1,K})$, the individual lives.

From the matrices $C_u(u = \overline{1,t})$ and $C^k(k = \overline{1,K})$, using the method of constructing membership functions based on pairwise comparisons [5], as described in section *III.B*, we find the degrees of membership of the location p_j to the criterion from the set G and form the rows of the matrix $S^k(k = \overline{1,K})$ from them:

$$S_k = \begin{matrix} \\ g_1 \\ g_2 \\ \vdots \\ g_t \\ r \end{matrix} \begin{matrix} p_1 & p_2 & \cdots & p_m \\ \pi_{S_k}(g_1,p_1) & \pi_{S_k}(g_1,p_2) & \cdots & \pi_{S_k}(g_1,p_m) \\ \pi_{S_k}(g_2,p_1) & \pi_{S_k}(g_2,p_2) & \cdots & \pi_{S_k}(g_2,p_m) \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{S_k}(g_t,p_1) & \pi_{S_k}(g_t,p_2) & \cdots & \pi_{S_k}(g_t,p_m) \\ \pi_{S_k}(r,p_1) & \pi_{S_k}(r,p_2) & \cdots & \pi_{S_k}(r,p_m) \end{matrix}, \quad (3.12)$$

where the rows $g_1, \dots, g_t$ correspond to the degrees of membership, found from the matrices $C_u(u = \overline{1,t})$, and the row r is found according to the same method from the corresponding matrices $C^k(k = \overline{1,K})$.

Let x_n^k be a population category x_n $(n = \overline{1,N})$, that lives in the residential complex H_k $(k = \overline{1,K})$. Let us find fuzzy sets E_j $(j = \overline{1,m})$, whose membership functions represent the location preferences p_j for the population category x_n^k. Then we find matrices T_k $(k = \overline{1,K})$, whose elements are the values of membership functions of these fuzzy sets $E_j, j = \overline{1,m}$:

$$T_k = \begin{matrix} \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \begin{matrix} p_1 & p_2 & \cdots & p_m \\ \mu_{E_1}(x_1^k,p_1) & \mu_{E_2}(x_1^k,p_2) & \cdots & \mu_{E_m}(x_1^k,p_m) \\ \mu_{E_1}(x_2^k,p_1) & \mu_{E_2}(x_2^k,p_2) & \cdots & \mu_{E_m}(x_2^k,p_m) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{E_1}(x_N^k,p_1) & \mu_{E_2}(x_N^k,p_2) & \cdots & \mu_{E_m}(x_N^k,p_m) \end{matrix}, \quad (3.13)$$

where

$$\mu_{E_j}(x_n^k, p_j) = \frac{\sum_{u=1}^{t+1} \Phi_R(x_n,g_u)\cdot \pi_{S_k}(g_u,p_j)}{\sum_{u=1}^{t+1} \Phi_R(x_n,g_u)}, g_{t+1} = r, n = \overline{1,N}, j = \overline{1,m}.$$

The function $\mu_{E_j}(x_n^k, p_j)$ shows the weighted degree of preference for location p_j by population category x_n^k.

From T_k we construct the matrices W_k $(k = \overline{1,K})$ in the following form:

$$W_k = \begin{pmatrix} \mu_{E_1\cap E_2}(x_1^k) & \mu_{E_1\cap E_3}(x_1^k) & \cdots & \mu_{E_{m-1}\cap E_m}(x_1^k) \\ \mu_{E_1\cap E_2}(x_2^k) & \mu_{E_1\cap E_3}(x_2^k) & \cdots & \mu_{E_{m-1}\cap E_m}(x_2^k) \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{E_1\cap E_2}(x_N^k) & \mu_{E_1\cap E_3}(x_N^k) & \cdots & \mu_{E_{m-1}\cap E_m}(x_N^k) \end{pmatrix}, \quad (3.14)$$

where element $\mu_{E_i\cap E_j}(x_n^k) = \min_{i<j} \left[\mu_{E_i}(x_n^k, p_i), \ \mu_{E_j}(x_n^k, p_j)\right]$, $i = \overline{1,m-2}, j = \overline{\iota+1,m}$.

Each fuzzy set E_j has an element of the universal set, on which the degree of membership $\mu_{E_j}(x_n^k, p_j)$ takes its maximum value, equal to $\max_{x_n^k}(\mu_{E_j}(x_n^k, p_j)$ (denote this value as Z_j^k). The intersection of the fuzzy sets $E_i \cap E_j$, defined by the membership function $\mu_{E_i\cap E_j}(x_n^k)$, will also have an element of the universal set, on which the membership degree $\mu_{E_i\cap E_j}(x_n^k)$ takes the its maximum value, equal to:

$$Z_{ij}^k = \max_{x_n^k} \ [\mu_{E_i\cap E_j}(x_n^k)], \ i = \overline{1,m-2}, j = \overline{\iota+1,m}.$$

Let us define the notion *of separability threshold* described in [6]. *The separability threshold* (denoted $\widetilde{l^k}$) – is the largest possible value from the corresponding matrix T_k (3.10), that does not exceed the number $\min_{i,j} Z_{ij}^k$. The separability threshold $\widetilde{l^k}$ has the following restriction:

$$\widetilde{l^k} < \min_{i,j} Z_{ij}^k, i = \overline{1,m-2}, j = \overline{\iota+1,m}.$$

From the corresponding matrix T_k (3.13), we search for the largest possible value, witch not exceeding the number $\min_{i,j} Z_{ij}^k$. After the choice of separability threshold has been made, we obtain the following level set (2.1):

$$M_j^k = \{x_n^k | \ \mu_{E_j}(x_n^k, p_j) \geq \widetilde{l^k}\}, \quad (3.15)$$

The level set M_j^k (1.1) allows us to infer how many people from the different population categories of the residential complex H_k $(k = \overline{1,K})$ favours location p_j out of the set of possible locations P at the separability threshold chosen, in the above described way $\widetilde{l^k}$.

We then compute the weighted degree of preference location p_j $(j = \overline{1,m})$ of individuals from the residential complex H_k $(k = \overline{1,K})$:

$$\chi_j^k = \frac{\sum_{x_n \in M_j^k} \Omega_{nk}\cdot \mu_{E_j}(x_n^k, p_j)}{\sum_{x_n^k} \Omega_{nk}}, \quad (3.16)$$

where Ω_{nk} is element of the matrix Ω (2.8) of settlement by residential complexes H_k $(k = \overline{1,K})$, M_j^k is level set (3.15).

As a result, we obtain the following vector of weighted degrees of preference of location options p_j:

$$V = \begin{matrix} p_1 & p_2 & \cdots & p_m \\ (v_1 & v_2 & \cdots & v_m) \end{matrix}, \quad (3.17)$$

where $v_1 = \sum_{k=1}^{K} \chi_j^k$, where χ_j^k is determined by formula (3.17).

Thus, the location option that has the highest degree of preference is considered the most convenient option for placing the new object, taking into account the preferences of all categories of the population in the microdistrict under consideration.

IV. Practical Realization

Necessary calculations for practical realization of the procedure of solving the problem were made using the mathematical package Mathcad 15, which was chosen due to the availability of additional functionality for working with fuzzy sets.

A. Problem statement

Let us consider a certain city microdistrict (Fig. 1), in which four possible locations for the metro station $P = \{p_1, \dots p_4\}$, (m = 4), have been proposed by the specialists based on their research. Locations p_2 and p_3 may have surface vestibules, while the others will be underground. There are also three residential complexes $H = \{H_1, \dots H_3\}$, (K = 3), in this microdistrict, along with shopping areas, a business center, two car parks, roads used by public transport, and three public transport stops. It is necessary to solve the decision-making problem of selecting the optimal location for the metro station, taking into account the attributes from a given set.

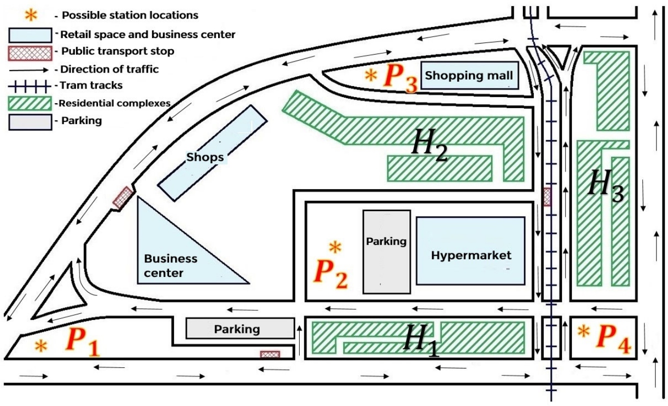

Fig. 1. Map of the microdistrict

Consider the attributes from the set $Y = \{y_1, \dots y_7\}$, (l = 7), according to which the optimal location for the new metro station will be chosen:

- y_1 – *Convenience* (the most convenient location of the new station for the population living in the microdistrict);
- y_2 – *Distance to surface* (km) (the shorter the distance from the underground lobby to the surface, the lower the station construction costs, making it the most favorable);
- y_3– *Pavilion costs* (bln RUB) (the presence of a building suitable for the construction of a ground lobby reduces the cost of building a new pavilion);
- y_4 – *Distance to the nearest land transport* (km) (the distance to the nearest public transport stop);
- y_5 – *Lobby capacity* (number of escalators working for entrance and exit, with the possibility of limiting the time of their operation for entrance; the more escalators are working, the greater the passenger flow at the station, and with increasing limitation of escalator operation time for entrance, the passenger flow decreases);
- y_6 – *Average area around the station for offices, retail space and car parks* (thousand m2) (a larger area guarantees an increase in the number of people using the new station);
- y_7 – *Distance to the center of gravity of passenger traffic* (km) (the closer the new station is to the area with the highest passenger traffic, the more unloaded the roads and pavements of the microdistrict will be, this attribute also takes into account the interests of residents from other parts of the city, who come to the considered microdistrict).

The attribute y_1 is not set directly in numerical terms, so it is necessary to calculate the vector of weighted degrees of preference for the proposed station locations for the «Convenience» attribute, as well as to determine the center of gravity of passenger flow to set distances for the attribute y_7.

B. Calculation of weighted degrees of preference for the attribute «Convenience»

This attribute is calculated for the population living directly in the given microdistrict, in residential complexes H_1, H_2, H_3 based on the method proposed in [6].

Let $X = \{x_1, \dots x_4\}$ – the set of population categories of the microdistrict, divided by the attributes: *age* and *frequency of metro usage*, each of which is considered as a fuzzy set with a corresponding membership function. Let us define:

x_1 – young and rarely used;

x_2 – young and frequent users;

x_3 – elderly and rarely used;

x_4 – elderly and frequent users.

Also given is the matrix $\Omega = (\Omega_{nk})_{n=\overline{1,4}}^{k=\overline{1,3}}$ (2.8) settlement by residential complexes, where the element Ω_{nk} is the number of people of population category x_n, $n = \overline{1,4}$, living in the residential complexes H_k, $k = \overline{1,3}$:

$$\Omega = \begin{matrix} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \begin{matrix} \begin{matrix} H_1 & H_2 & H_3 \end{matrix} \\ \begin{pmatrix} 510 & 431 & 420 \\ 294 & 387 & 264 \\ 205 & 125 & 272 \\ 231 & 153 & 128 \end{pmatrix} \end{matrix}.$$

Let the following set of criteria $G = \{g_1, \dots g_4, r\}$, where:

g_1 – access to the station by public transport,

g_2 – availability of shops and retail space near the station,

g_3 – availability of car parks near the station,

g_4 – location of the ground lobby (on the surface or underground),

r – distance to residential complexes (straight line).

Using Saaty's nine-point scale [3], set the consistent pairwise comparison matrices B_n (2.9) for each population category x_n, $n = \overline{1,4}$, (the “>” sign denotes the advantage of one criterion over another):

for x_1: $g_3 > g_2 > g_1 > r > g_4$

$$B_1 = \begin{matrix} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ r \end{matrix} \begin{matrix} \begin{matrix} g_1 & g_2 & g_3 & g_4 & r \end{matrix} \\ \begin{pmatrix} 1 & 1/4 & 1/5 & 3 & 3 \\ 4 & 1 & 1/4 & 6 & 5 \\ 5 & 4 & 1 & 9 & 9 \\ 1/3 & 1/6 & 1/9 & 1 & 1/3 \\ 1/3 & 1/6 & 1/9 & 3 & 1 \end{pmatrix} \end{matrix};$$

for x_2: $g_1 > g_2 > g_3 > g_4 > r$

$$B_2 = \begin{matrix} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ r \end{matrix} \begin{matrix} \begin{matrix} g_1 & g_2 & g_3 & g_4 & r \end{matrix} \\ \begin{pmatrix} 1 & 4 & 5 & 8 & 8 \\ 1/4 & 1 & 4 & 5 & 5 \\ 1/5 & 1/4 & 1 & 4 & 4 \\ 1/8 & 1/5 & 1/4 & 1 & 3 \\ 1/8 & 1/5 & 1/4 & 1/3 & 1 \end{pmatrix} \end{matrix};$$

for x_3: $g_1 > g_2 > r > g_3 > g_4$

$$B_3 = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ r \end{array} \begin{array}{c} \begin{matrix} g_1 & g_2 & g_3 & g_4 & r \end{matrix} \\ \begin{pmatrix} 1 & 5 & 6 & 7 & 5 \\ 1/5 & 1 & 4 & 5 & 3 \\ 1/6 & 1/4 & 1 & 4 & 1/4 \\ 1/7 & 1/5 & 1/4 & 1 & 1/4 \\ 1/5 & 1/3 & 4 & 4 & 1 \end{pmatrix} \end{array};$$

for x_4: $g_4 > g_1 > r > g_2 > g_3$

$$B_4 = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ r \end{array} \begin{array}{c} \begin{matrix} g_1 & g_2 & g_3 & g_4 & r \end{matrix} \\ \begin{pmatrix} 1 & 4 & 5 & 1/3 & 3 \\ 1/4 & 1 & 4 & 1/5 & 1/4 \\ 1/5 & 1/4 & 1 & 1/6 & 1/4 \\ 3 & 5 & 6 & 1 & 5 \\ 1/3 & 4 & 4 & 1/5 & 1 \end{pmatrix} \end{array}.$$

We find the eigenvectors of matrices B_n, $n = \overline{1,4}$, corresponding to their maximum eigenvalues and write them as rows of matrix R (3.10) (fuzzy binary relation matrix):

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{matrix} g_1 & g_2 & g_3 & g_4 & r \end{matrix} \\ \begin{pmatrix} 0.172 & 0.403 & 0.892 & 0.059 & 0.096 \\ 0.881 & 0.410 & 0.204 & 0.096 & 0.062 \\ 0.895 & 0.361 & 0.120 & 0.063 & 0.225 \\ 0.444 & 0.142 & 0.073 & 0.842 & 0.262 \end{pmatrix} \end{array}.$$

Each element of the matrix R shows the relative degree of advantage of the criteria for the given categories of the population, i.e., a higher value in the row corresponds to the most important criterion.

Suppose that, based on the nine-point Saaty scale [3], expert judgments of criteria g_1, g_2, g_3, g_4, r for each proposed metro station location p_1, p_2, p_3, p_4 were obtained as the following matrix of consistent pairwise comparisons C_p ($p = \overline{1,4}$) (3.11):

for the criterion «*Access to the station by public transport*» g_1: $p_4 > p_1 > p_3 > p_2$

$$C_1 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1 & 5 & 4 & 1/4 \\ 1/5 & 1 & 1/4 & 1/6 \\ 1/4 & 4 & 1 & 1/5 \\ 4 & 6 & 5 & 1 \end{pmatrix} \end{array};$$

for the criterion «*Availability of shops and retail space near the station*» g_2: $p_2 > p_3 > p_4 > p_1$

$$C_2 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1 & 1/6 & 1/5 & 1/5 \\ 6 & 1 & 4 & 4 \\ 5 & 1/4 & 1 & 4 \\ 5 & 1/4 & 1/4 & 1 \end{pmatrix} \end{array};$$

for the criterion «*Availability car parks near the station*» g_3: $p_1 > p_2 > p_4 > p_3$

$$C_3 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1 & 3 & 7 & 6 \\ 1/3 & 1 & 6 & 5 \\ 1/7 & 1/6 & 1 & 1/4 \\ 1/6 & 1/5 & 4 & 1 \end{pmatrix} \end{array};$$

for the criterion «*Location of the ground lobby on the surface or underground*» it is assumed that at locations p_2 and p_3 it is possible to build a lobby underground, and at p_1, p_4 – only on the surface. At the same time, the possibility of building the lobby underground significantly reduces the station construction costs, as the construction of a lobby on the surface requires additional measures to combat vibration and noise from escalators and requires space for a pavilion. The depth of the underground lobby also affects the cost:

g_4: $p_3 > p_2 > p_1 > p_4$

$$C_4 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1 & 1/5 & 1/5 & 3 \\ 5 & 1 & 1/3 & 5 \\ 5 & 3 & 1 & 5 \\ 1/3 & 1/5 & 1/5 & 1 \end{pmatrix} \end{array}.$$

Let us also define expert judgements for the criterion r: «Distance to residential complexes» as consistent matrices of pairwise comparisons C^k (2.11), where the index k corresponds to residential complexes H_k, $k = \overline{1,3}$:

$$C^1 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1 & 1/4 & 5 & 1/4 \\ 4 & 1 & 5 & 4 \\ 1/5 & 1/5 & 1 & 1/5 \\ 4 & 1/4 & 5 & 1 \end{pmatrix} \end{array}$$

$$C^2 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1 & 1/4 & 1/6 & 3 \\ 4 & 1 & 1/4 & 5 \\ 6 & 4 & 1 & 7 \\ 1/3 & 1/5 & 7 & 1 \end{pmatrix} \end{array}$$

$$C^3 = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 1 & 1/4 & 1/4 & 1/7 \\ 4 & 1 & 1/3 & 1/4 \\ 4 & 3 & 1 & 1/4 \\ 7 & 4 & 4 & 1 \end{pmatrix} \end{array}$$

After that we find matrices S_k (3.12), whose rows correspond to fuzzy sets matched to criteria from the set of G.

$$S_1 = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ r \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 0.403 & 0.082 & 0.180 & 0.894 \\ 0.079 & 0.877 & 0.424 & 0.210 \\ 0.868 & 0.464 & 0.073 & 0.161 \\ 0.170 & 0.490 & 0.849 & 0.098 \\ 0.217 & 0.870 & 0.087 & 0.435 \end{pmatrix} \end{array},$$

$$S_2 = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ r \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 0.403 & 0.082 & 0.180 & 0.894 \\ 0.079 & 0.877 & 0.424 & 0.210 \\ 0.868 & 0.464 & 0.073 & 0.161 \\ 0.170 & 0.490 & 0.849 & 0.098 \\ 0.149 & 0.376 & 0.911 & 0.079 \end{pmatrix} \end{array},$$

$$S_3 = \begin{array}{c} \\ g_1 \\ g_2 \\ g_3 \\ g_4 \\ r \end{array} \begin{array}{c} \begin{matrix} p_1 & p_2 & p_3 & p_4 \end{matrix} \\ \begin{pmatrix} 0.403 & 0.082 & 0.180 & 0.894 \\ 0.079 & 0.877 & 0.424 & 0.210 \\ 0.868 & 0.464 & 0.073 & 0.161 \\ 0.170 & 0.490 & 0.849 & 0.098 \\ 0.085 & 0.212 & 0.370 & 0.901 \end{pmatrix} \end{array}.$$

Using (3.13), we find the matrices T_k, $k = \overline{1,3}$, then we get:

$$T_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{cccc} p_1 & p_2 & p_3 & p_4 \\ \hline \end{array}$$

$$T_1 = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \overset{\begin{matrix} p_1 & \;\; p_2 & \;\; p_3 & \;\; p_4 \end{matrix}}{\begin{pmatrix} 0.559 & 0.551 & 0.201 & 0.265 \\ 0.360 & 0.380 & 0.263 & 0.570 \\ 0.332 & 0.404 & 0.238 & 0.601 \\ 0.257 & 0.474 & 0.501 & 0.360 \end{pmatrix}},$$

$$T_2 = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \overset{\begin{matrix} p_1 & \;\; p_2 & \;\; p_3 & \;\; p_4 \end{matrix}}{\begin{pmatrix} 0.555 & 0.522 & 0.249 & 0.244 \\ 0.357 & 0.361 & 0.294 & 0.557 \\ 0.323 & 0.337 & 0.349 & 0.552 \\ 0.247 & 0.400 & 0.623 & 0.307 \end{pmatrix}},$$

$$T_3 = \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \overset{\begin{matrix} p_1 & \;\; p_2 & \;\; p_3 & \;\; p_4 \end{matrix}}{\begin{pmatrix} 0.551 & 0.512 & 0.217 & 0.292 \\ 0.355 & 0.355 & 0.273 & 0.588 \\ 0.314 & 0.315 & 0.276 & 0.664 \\ 0.238 & 0.376 & 0.543 & 0.429 \end{pmatrix}}.$$

From matrices T_k on the basis of (3.14), we construct matrices W_k, $k = \overline{1,3}$:

$$W_1 = \begin{pmatrix} 0.551 & 0.201 & 0.265 & 0.201 & 0.265 & 0.201 \\ 0.360 & 0.263 & 0.360 & 0.263 & 0.380 & 0.263 \\ 0.332 & 0.238 & 0.332 & 0.238 & 0.404 & 0.238 \\ 0.257 & 0.257 & 0.257 & 0.474 & 0.360 & 0.360 \end{pmatrix},$$

$$W_2 = \begin{pmatrix} 0.522 & 0.249 & 0.244 & 0.249 & 0.244 & 0.244 \\ 0.357 & 0.294 & 0.357 & 0.224 & 0.361 & 0.294 \\ 0.323 & 0.323 & 0.332 & 0.337 & 0.337 & 0.349 \\ 0.247 & 0.247 & 0.247 & 0.400 & 0.307 & 0.307 \end{pmatrix},$$

$$W_3 = \begin{pmatrix} 0.512 & 0.217 & 0.292 & 0.217 & 0.292 & 0.217 \\ 0.355 & 0.273 & 0.355 & 0.273 & 0.355 & 0.273 \\ 0.314 & 0.276 & 0.314 & 0.276 & 0.315 & 0.276 \\ 0.238 & 0.238 & 0.238 & 0.376 & 0.376 & 0.429 \end{pmatrix}.$$

Then we calculate the separability threshold $\widetilde{l^k}$. For matrices W_k, $k = \overline{1,3}$, we get the following results:

for W_1: $Z^1_{12} = 0.551; Z^1_{13} = 0.263; Z^1_{14} = 0.360; Z^1_{23} = 0.474; Z^1_{24} = 0.404; Z^1_{34} = 0.360.$

for W_2: $Z^2_{12} = 0.522; Z^2_{13} = 0.323; Z^2_{14} = 0.357; Z^2_{23} = 0.400; Z^2_{24} = 0.361; Z^2_{34} = 0.349.$

for W_3: $Z^3_{12} = 0.512; Z^3_{13} = 0.276; Z^3_{14} = 0.355; Z^3_{23} = 0.376; Z^3_{24} = 0.376; Z^3_{34} = 0.429.$

For each W_k we define $\min\limits_{i<j} Z^k_{ij}$, $(i = \overline{1,3}, j = \overline{2,4}, k = \overline{1,3})$:

$$\min Z^1 = 0.263; \min Z^2 = 0.323; \min Z^3 = 0.276.$$

From the corresponding matrices T_k we find the largest possible value not exceeding $\min Z^k$, $k = \overline{1,3}$ respectively:

$$\widetilde{l^1} = 0.257; \widetilde{l^2} = 0.307; \widetilde{l^3} = 0.273.$$

Based on (3.15) we obtain level sets M^k_m, $k = \overline{1,3}$, $m = \overline{1,4}$:

$$M^1_1 = \{x_1, x_2, x_3, x_4\}, M^1_2 = \{x_1, x_2, x_3, x_4\}, M^1_3 = \{x_2, x_4\}, M^1_4 = \{x_1, x_2, x_3, x_4\};$$

$$M^2_1 = \{x_1, x_2, x_3\}, M^2_2 = \{x_1, x_2, x_3, x_4\}, M^2_3 = \{x_3, x_4\}, M^2_4 = \{x_2, x_3, x_4\};$$

$$M^3_1 = \{x_1, x_2, x_3\}, M^3_2 = \{x_1, x_2, x_3, x_4\}, M^3_3 = \{x_2, x_3, x_4\}, M^3_4 = \{x_1, x_2, x_3, x_4\}.$$

We calculate the weighted degree of preference χ^k_m of the population from the residential complexes H_k, $k = \overline{1,3}$ of the proposed location p_m, $m = \overline{1,4}$, using formula (3.16):

$$\chi^1_1 = 0.418; \chi^1_2 = 0.472; \chi^1_3 = 0.156; \chi^1_4 = 0.411;$$

$$\chi^2_1 = 0.381; \chi^2_2 = 0.427; \chi^2_3 = 0.127; \chi^2_4 = 0.302;$$

$$\chi^3_1 = 0.379; \chi^3_2 = 0.408; \chi^3_3 = 0.200; \chi^3_4 = 0.474.$$

Based on (3.17), we obtain the required vector of weighted degrees of preference for the proposed station locations p_m, $m = \overline{1,4}$ with respect to the «*Convenience*» feature, for all categories of the population living in residential complexes of the microdistrict:

$$V = \overset{\begin{matrix} p_1 & \;\; p_2 & \;\; p_3 & \;\; p_4 \end{matrix}}{\begin{pmatrix} 1.178 & 1.307 & 0.483 & 1.187 \end{pmatrix}}.$$

C. Calculating the center of gravity of passenger traffic

In order to make the selection of metro station location p_m $(m = \overline{1,4})$ more optimal, it is assumed that the station should be located as close as possible to the areas with the highest passenger flow. Let's use the method of determining the center of gravity from [7] and calculate *the center of gravity of passenger flow*.

Let us assume that, in the given city microdistrict, there is a developed road network, and with the help of installed cameras on the roads at each proposed station p_m the average passenger traffic Γ_m $(m = \overline{1,4})$ (thousands of people/day) was recorded.

$$\Gamma_1 = 15, \Gamma_2 = 5, \Gamma_3 = 11, \Gamma_4 = 21.$$

For this section of the city, we introduce a coordinate system with $\tilde{X}$ and $\tilde{Y}$ axes (unit of measurement: metre). Let's transfer the contour of the considered microdistrict to it and determine the coordinates of the station locations:

$(\widetilde{x_1}, \widetilde{y_1}) = (93, 21)$, $(\widetilde{x_2}, \widetilde{y_2}) = (260, 140)$, $(\widetilde{x_3}, \widetilde{y_3}) = (350, 290)$, $(\widetilde{x_4}, \widetilde{y_4}) = (550, 30)$.

Applying formula (2.7), we calculate the coordinates of the location of the desired center of gravity of passenger flow (Fig. 2):

$$(\tilde{X}_{center}, \tilde{Y}_{center}) = (348, 93).$$

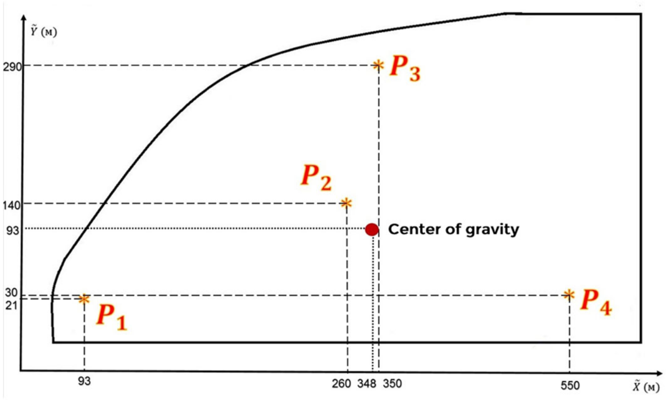

Fig. 2. Determination of the center of gravity of passenger flow

Thus, station № 2 is the closest to the center of gravity, i.e., the place where passenger flows are the highest.

D. Choose the optimal location of the metro station

Using the method of fuzzy multi-criteria analysis of options based on pairwise comparisons from [5] and the Bellman-Zadeh scheme from [8], which allows us to determine the best option, we will choose the optimal location of the metro station.

When choosing the optimal station location $P = \{p_1, \dots p_4\}$, $(m = 4)$ of a metro station, it is necessary to analyze the variants of the given locations and the attributes $Y = \{y_1, \dots y_7\}$, $(l = 7)$, that characterize them. Table II below lists the attributes and their values for each possible station location.

TABLE II. ATTRIBUTES VALUES FOR THE ASSESSMENT OF STATION LOCATIONS

Y	Attributes	p_1	p_2	p_3	p_4
y_1	Convenience	1.178	1.307	0.483	1.187
y_2	Distance to surface (km)	0.02	0.034	0.03	0.015
y_3	Pavilion costs (RUB billion)	4.2	3.15	3.05	3.9
y_4	Distance to the nearest land transport (km)	0.28	0.55	0.3	0.24
y_5	Capacity of the station lobby*	2	3	2	4
y_6	Average area around the station for offices, retail space and car parks (thousand m2)	20	66	15	27
y_7	Distance to the center of gravity of passenger traffic (km)**	0.26	0.088	0.22	0.20

*— p_1: 1 escalator for entrance (opening hours: 6:00-10:00), 1 escalator for exit;

— p_2: 2 escalators for entrance (opening hours: 6:00-10:00), 1 escalator for exit;

— p_3: 1 escalator to entrance, 1 escalator to exit;

— p_4: 2 escalators for entrance, 2 escalators for exit;

** — calculated from the coordinates obtained from Sect. IV.C.

E. Matrices of pairwise comparisons

Using expert judgments based on the nine-point Saaty scale [3], we construct the following matrix of consistent pairwise comparisons Q_l for each attribute y_l, $l = \overline{1,7}$.

- for the attribute «Convenience»:

$$Q_1 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ p_1 & 1 & 1/5 & 6 & 1/4 \\ p_2 & 5 & 1 & 7 & 4 \\ p_3 & 1/6 & 1/7 & 1 & 1/7 \\ p_4 & 4 & 1/4 & 7 & 1 \end{matrix};$$

- for the attribute «Distance to surface»:

$$Q_2 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ p_1 & 1 & 7 & 6 & 1/4 \\ p_2 & 1/7 & 1 & 1/4 & 1/7 \\ p_3 & 1/6 & 4 & 1 & 1/6 \\ p_4 & 4 & 7 & 6 & 1 \end{matrix};$$

- for the attribute «Pavilion costs»:

$$Q_3 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ p_1 & 1 & 1/6 & 1/6 & 1/4 \\ p_2 & 6 & 1 & 1/3 & 5 \\ p_3 & 6 & 3 & 1 & 6 \\ p_4 & 4 & 1/5 & 1/6 & 1 \end{matrix};$$

- for the attribute «Distance to the nearest land transport»:

$$Q_4 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ p_1 & 1 & 5 & 3 & 1/3 \\ p_2 & 1/5 & 1 & 1/4 & 1/7 \\ p_3 & 1/3 & 4 & 1 & 1/3 \\ p_4 & 3 & 7 & 3 & 1 \end{matrix};$$

- for the attribute «Capacity of the station lobby»:

$$Q_5 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ p_1 & 1 & 1/5 & 1/4 & 1/7 \\ p_2 & 5 & 1 & 4 & 1/4 \\ p_3 & 4 & 1/4 & 1 & 1/7 \\ p_4 & 7 & 4 & 7 & 1 \end{matrix};$$

- for the attribute «Average area around the station for offices, retail space and car parks»:

$$Q_6 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ p_1 & 1 & 1/7 & 3 & 1/3 \\ p_2 & 7 & 1 & 7 & 5 \\ p_3 & 1/3 & 1/7 & 1 & 1/3 \\ p_4 & 3 & 1/5 & 3 & 1 \end{matrix};$$

- for the attribute «Distance to the center of gravity of passenger traffic»:

$$Q_7 = \begin{matrix} & p_1 & p_2 & p_3 & p_4 \\ p_1 & 1 & 1/7 & 1/3 & 1/5 \\ p_2 & 7 & 1 & 7 & 6 \\ p_3 & 3 & 1/7 & 1 & 1/4 \\ p_4 & 5 & 1/6 & 4 & 1 \end{matrix}.$$

For each attribute y_l, $l = \overline{1,7}$, we construct fuzzy sets $\widetilde{Q_l}$, $i = \overline{1,7}$, using the expression (3.2):

$$\widetilde{Q_1} = \left\{\frac{0.123}{p_1}, \frac{\boxed{0.570}}{p_2}, \frac{0.041}{p_3}, \frac{0.266}{p_4}\right\}; \widetilde{Q_2} = \left\{\frac{0.287}{p_1}, \frac{0.043}{p_2}, \frac{0.091}{p_3}, \frac{\boxed{0.579}}{p_4}\right\};$$

$$\widetilde{Q_3} = \left\{\frac{0.050}{p_1}, \frac{0.300}{p_2}, \frac{\boxed{0.546}}{p_3}, \frac{0.105}{p_4}\right\}; \widetilde{Q_4} = \left\{\frac{0.278}{p_1}, \frac{0.053}{p_2}, \frac{0.150}{p_3}, \frac{\boxed{0.519}}{p_4}\right\}; \widetilde{Q_5} = \left\{\frac{0.048}{p_1}, \frac{0.242}{p_2}, \frac{0.102}{p_3}, \frac{\boxed{0.608}}{p_4}\right\};$$

$$\widetilde{Q_6} = \left\{\frac{0.103}{p_1}, \frac{\boxed{0.647}}{p_2}, \frac{0.059}{p_3}, \frac{0.191}{p_4}\right\}; \widetilde{Q_7} = \left\{\frac{0.049}{p_1}, \frac{\boxed{0.655}}{p_2}, \frac{0.087}{p_3}, \frac{0.209}{p_4}\right\}.$$

We can now conclude that station p_2 has the highest weight for attributes y_1, y_6, y_7, meaning that the location of station p_2 is the most favorable for these features. Station p_4 is the best option for attributes y_2, y_4, y_5, while station p_3 is optimal for attribute y_3.

F. Case of equilibrium and non-equilibrium attributes

Using expression (3.4), we get the following fuzzy set $\tilde{G}$ for equilibrium features, defined as the intersection of all fuzzy sets $\widetilde{Q_l}$, $i = \overline{1,7}$,:

$$\tilde{G} = \widetilde{Q_1} \cap \ldots \cap \widetilde{Q_7} = \left\{\frac{0.048}{p_1}, \frac{0.043}{p_2}, \frac{0.041}{p_3}, \frac{\boxed{0.105}}{p_4}\right\},$$

which shows that p_4 has the highest degree of membership, i.e., the location of station №4 has a significant advantage over the others with respect to all attributes simultaneously.

In the case of non-equilibrium attributes, it is necessary to find the optimal station location that will be the best for all given attributes y_1 ... y_7, taking into account the importance of each.

Let us calculate the weights using the method proposed in [15] using the order scale (a normalized ranking scale). On a scale from 1 to 5, let us evaluate the attributes y_l: y_1 – 5; y_2 – 1; y_3 – 2; y_4 – 3; y_5 – 4; y_6 – 4; y_7 – 3. Using formula (2.5) we get the following weights (Table III):

TABLE III. VALUES OF WEIGHTS FOR ATTRIBUTES

	Attribute	Weight
y_1	Convenience	0.277
y_2	Distance to surface	0.045
y_3	Pavilion costs	0.091
y_4	Distance to the nearest land transport (km)	0.136
y_5	Capacity of the station lobby	0.182
y_6	Average area around the station for offices, retail space and car parks	0.182
y_7	Distance to the center of gravity of passenger traffic	0.136

Next, in order to find the fuzzy set $\tilde{G}$, we use (3.6)

$$\widetilde{Q_1} = \left\{\frac{0.123^{0.227}}{p_1}, \frac{0.570^{0.227}}{p_2}, \frac{0.041^{0.227}}{p_3}, \frac{0.266^{0.227}}{p_4}\right\} = \left\{\frac{0.621}{p_1}, \frac{0.880}{p_2}, \frac{0.484}{p_3}, \frac{0.740}{p_4}\right\};$$

$$\widetilde{Q_2} = \left\{\frac{0.287^{0.045}}{p_1}, \frac{0.043^{0.045}}{p_2}, \frac{0.091^{0.045}}{p_3}, \frac{0.579^{0.045}}{p_4}\right\} = \left\{\frac{0.945}{p_1}, \frac{0.868}{p_2}, \frac{0.898}{p_3}, \frac{0.946}{p_4}\right\};$$

$$\widetilde{Q_3} = \left\{\frac{0.050^{0.091}}{p_1}, \frac{0.300^{0.091}}{p_2}, \frac{0{,}546^{0.091}}{p_3}, \frac{0.105^{0.091}}{p_4}\right\} = \left\{\frac{0.761}{p_1}, \frac{0.896}{p_2}, \frac{0.946}{p_3}, \frac{0.815}{p_4}\right\};$$

$$\widetilde{Q_4} = \left\{\frac{0.278^{0.136}}{p_1}, \frac{0.053^{0.136}}{p_2}, \frac{0.150^{0.136}}{p_3}, \frac{0.519^{0.136}}{p_4}\right\} = \left\{\frac{0.840}{p_1}, \frac{0.671}{p_2}, \frac{0.773}{p_3}, \frac{0.915}{p_4}\right\};$$

$$\widetilde{Q_5} = \left\{\frac{0.048^{0.182}}{p_1}, \frac{0.242^{0.182}}{p_2}, \frac{0.102^{0.182}}{p_3}, \frac{0.608^{0.182}}{p_4}\right\} = \left\{\frac{0.575}{p_1}, \frac{0.772}{p_2}, \frac{0.660}{p_3}, \frac{0.913}{p_4}\right\};$$

$$\widetilde{Q_6} = \left\{\frac{0.103^{0.182}}{p_1}, \frac{0.647^{0.182}}{p_2}, \frac{0.059^{0.182}}{p_3}, \frac{0.191^{0.182}}{p_4}\right\} = \left\{\frac{0.661}{p_1}, \frac{0.924}{p_2}, \frac{0.597}{p_3}, \frac{0.740}{p_4}\right\};$$

$$\widetilde{Q_7} = \left\{\frac{0.049^{0.136}}{p_1}, \frac{0.655^{0.136}}{p_2}, \frac{0.087^{0.136}}{p_3}, \frac{0.209^{0.136}}{p_4}\right\} = \left\{\frac{0.664}{p_1}, \frac{0.944}{p_2}, \frac{0.717}{p_3}, \frac{0.808}{p_4}\right\}.$$

Hence, we obtain the degrees of membership $\mu_{\widetilde{Q_l}}(p_m)$, $m = \overline{1,4}$, for computing the fuzzy set $\tilde{G}$ in the following form:

$$\begin{aligned}
\mu_{\widetilde{Q_l}}(p_1) &= (0.621, 0.945, 0.761, 0.84, 0.575, 0.661, 0.664);\\
\mu_{\widetilde{Q_l}}(p_2) &= (0.880, 0.868, 0.896, 0.671, 0.772, 0.924, 0.944);\\
\mu_{\widetilde{Q_l}}(p_3) &= (0.484, 0.898, 0.946, 0.773, 0.660, 0.597, 0.717);\\
\mu_{\widetilde{Q_l}}(p_4) &= (0.740, 0.976, 0.815, 0.915, 0.913, 0.740, 0.808).
\end{aligned} \tag{4.1}$$

Let's define $\min_{i=\overline{1,7}} \mu_{\widetilde{Q_l}}(p_m), m = \overline{1,4}$:

$$\min_{i=\overline{1,7}} \mu_{\widetilde{Q_l}}(p_1) = 0.575,\ \min_{i=\overline{1,7}} \mu_{\widetilde{Q_l}}(p_2) = 0.671,\ \min_{i=\overline{1,7}} \mu_{\widetilde{Q_l}}(p_3) = 0.484,\ \min_{i=\overline{1,7}} \mu_{\widetilde{Q_l}}(p_4) = 0.740.$$

As a result, the fuzzy set $\tilde{G}$ has the form:

$$\tilde{G} = \left\{\frac{0.575}{p_1}, \frac{0.671}{p_2}, \frac{0.484}{p_3}, \frac{0.740}{p_4}\right\}. \tag{4.2}$$

From (4.2) we observe that the highest value of the membership function corresponds to p_4. Therefore, for the given city microdistrict, station №4 is the optimal location simultaneously for all attributes, taking into account the relative importance of each. Let us define fuzzy sets representing the correspondence of the proposed stations p_1, p_2, p_3, p_4 to the attributes $y_1, \ldots y_7$, using the membership functions defined in (4.1) and plot these functions (Fig.3):

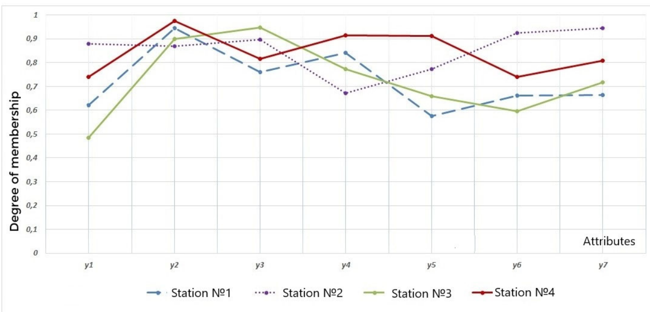

Fig. 3. Comparison of station location options taking into account the importance of attributes

Fig. 3 illustrates the advantage of location №4 for the new metro station, taking into account the relative importance of each attribute y_l, $l = \overline{1,7}$.

Having analyzed the obtained results, it is possible to formulate several recommendations that may be used during the construction of a new metro station in order to improve the selected location according to certain attributes from the set Y. To reduce the pavilion construction costs (attribute y_3) for station №4, it is suggested to build not 4 escalators for entrance and exit, as initially assumed, but, for example, 2 escalators for entrance and 1 for exit. Reducing the number of escalators would significantly lower construction expenses, which is economically beneficial. Part of the saved funds could be used to build a shop or a shopping center near the proposed station location, which would make Location №4 even more convenient for the population living in the microdistrict, as well as improve indicators for attribute y_6.

V. Conclusion

When solving the problem of selecting the location of a new metro station, that is most convenient for the population living in the microdistrict, the following methods and approaches were combined: the decision-making approach based on fuzzy sets for formalizing preferences, the method of multi-criteria analysis of options in the case of equilibrium and non-equilibrium attributes according to the Bellman-Zadeh scheme, the method for determining the center of gravity of the physical distribution model. This combination of methods, as well as similar ones, can be applied in various fields such as medicine, economics, education and others. For example, it is possible to choose the location of schools, shops, hospitals and other social infrastructure in order to improve the quality of life of the population. When solving such problems, particularly in constructing matrices of pairwise comparisons, it is not necessary to rely solely on the Saaty scale, other established comparison scales may also be used [16].

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